

A NOTE ON COMMUTATIVE ALGEBRAS AND THEIR MODULES IN QUASICATEGORIES

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The purpose of this document is to develop a neat combinatorial theory of modules over commutative algebras in ∞ -categories in the vein of the theory of modules over associative algebras set out in [Lur12, §4.2, 4.3]. In fact, we'll describe a cute commutative analog \mathcal{CM}^\otimes of the operads \mathcal{LM}^\otimes and \mathcal{BM}^\otimes of [Lur12, §4.2.1] and [Lur12, §4.3.1]. This should ease the task of constructing and manipulating such modules. In particular, we prove the following theorem, which plays a role in [Gla14]. It's tautological for 1-categories but sort of subtle for ∞ -categories, and it's just generally nice to know:

Theorem 1. Let \mathbf{C}^\otimes be a symmetric monoidal ∞ -category. We'll denote the category of finite sets by \mathcal{F} and the category of finite pointed sets by \mathcal{F}_* . A datum comprising a commutative algebra E in \mathbf{C} and a module M over it - that is, an object of $\mathbf{Mod}^{\mathcal{F}_*}(\mathbf{C})$, the underlying ∞ -category of Lurie's ∞ -operad $\mathbf{Mod}^{\mathcal{F}_*}(\mathbf{C})^\otimes$ [Lur12, Definition 3.3.3.8] - gives rise functorially to a functor

$$A_{E,M} : \mathcal{F}_* \rightarrow \mathbf{C}$$

such that

$$A_{E,M}(S) \simeq E^{\otimes S^o} \otimes M.$$

We'll prove Theorem 1 by first describing the ∞ -operad \mathcal{CM}^\otimes that parametrizes this data and then giving a map from \mathcal{F}_* to the symmetric monoidal envelope of \mathcal{CM}^\otimes . We'll work extensively with the model category of ∞ -preoperads [Lur12, §2.1.4], which we'll denote \mathcal{PO} .

Definition 2. We define the operad \mathcal{CM}^\otimes as (the nerve of) the 1-category in which

- an object is a pair (S, U) consisting of an object S of \mathcal{F}_* and a subset U of S^o ;
- a morphism from (S, U) to (T, V) is a morphism $f : S \rightarrow T$ in \mathcal{F}_* such that for each $v \in V$, the set $U \cap f^{-1}(v)$ has cardinality exactly 1.

It's easily checked that the functor $\mathcal{CM}^\otimes \rightarrow \mathcal{F}_*$ that maps (S, U) to S makes \mathcal{CM}^\otimes into an ∞ -operad. The following result isolates the hard work involved in proving Theorem 1:

Proposition 3. We give $(\mathcal{F}_*)_{\langle 1 \rangle /}$ the structure of an ∞ -preoperad by letting the target map $t : (\mathcal{F}_*)_{\langle 1 \rangle /} \rightarrow \mathcal{F}_*$ create marked edges. Define a map of ∞ -preoperads

$$\phi : (\mathcal{F}_*)_{\langle 1 \rangle /} \rightarrow \mathcal{CM}^\otimes$$

by

$$\phi(j : \langle 1 \rangle \rightarrow S) = \begin{cases} (S, \{j(1)\}) & \text{if } j(1) \in S^o \\ (S, \emptyset) & \text{otherwise.} \end{cases}$$

Thus we embed $(\mathcal{F}_*)_{\langle 1 \rangle /}$ as the full subcategory of \mathcal{CM}^\otimes spanned by those objects (S, U) for which the cardinality of U is at most 1.

Then ϕ is an trivial cofibration in \mathcal{PO} .

The proof will take the form of a series of lemmas.

Lemma 4. Suppose $q : \mathbf{E} \rightarrow \mathbf{B}$ is an inner fibration of ∞ -categories, K a simplicial set and $r : K^\triangleleft \rightarrow \mathbf{B}$ a map such that for each edge $e : k_1 \rightarrow k_2$ of K and for each $l \in \mathbf{E}$ with $q(l) = r(k_1)$, there is a cocartesian lift of e to \mathbf{E} with source l . Denote the cone point of K^\triangleleft by c and suppose $d \in \mathbf{E}$ is such that $q(d) = r(c)$. Then there is a map $r' : K^\triangleleft \rightarrow \mathbf{E}$ lifting r and taking every edge of K^\triangleleft to a cocartesian edge of \mathbf{E} .

Proof. Clearly we can lift in such a way that the image of every edge of K^\triangleleft with source c is cocartesian; let r' be such a lift. We claim that r' already has the desired property. Indeed, r' can be viewed as a section of the cocartesian fibration $q_{K^\triangleleft} : \mathbf{E} \times_{\mathbf{B}} K^\triangleleft \rightarrow K^\triangleleft$, and then the result follows from [Lur09, Proposition 2.4.2.7]. \square

Lemma 5. Let $p : \mathcal{O}^\otimes \rightarrow \mathcal{F}_*$ be any ∞ -operad. For any set T , let \mathcal{P}_T be the poset of subsets of T ordered by reverse inclusion, and let $\mathcal{P}'_T = \mathcal{P}_T \setminus \{T\}$, so that $\mathcal{P}_T \cong (\mathcal{P}'_T)^\triangleleft$. For each $S \in \mathcal{F}_*$, let $r_S : \mathcal{P}_{S^o} \rightarrow \mathcal{F}_*$ denote the obvious diagram of inert morphisms. Let $X \in \mathcal{O}^\otimes$, let $\rho : \mathcal{P}_{S^o} \rightarrow \mathcal{O}^\otimes$ be the cocartesian lift of r_S with $\rho(S^o) = X$ whose existence is guaranteed by Lemma 4, and suppose $|S^o| > 1$. Then ρ is a limit diagram relative to p .

Proof. We work by induction on the size of S^o ; the case $|S^o| = 2$ is an immediate consequence of the ∞ -operad axioms. For each $k \in \mathbb{N}$, let $\mathcal{P}_T^{\leq k}$ denote the poset of subsets of T of cardinality at most k . Then the restriction of ρ to $(\mathcal{P}_{S^o}^{\leq 1})^\triangleleft$ is a p -limit diagram by the ∞ -operad axioms. We now argue by induction on k that for each k with $1 \leq k \leq |S^o| - 1$, the restriction of ρ to $(\mathcal{P}_{S^o}^{\leq k})^\triangleleft$ is a limit diagram. Indeed, for each such $k > 1$, the $|S^o| = k$ edition of the lemma implies that $\rho|_{\mathcal{P}_{S^o}^{\leq k}}$ is p -right Kan extended from $\rho|_{\mathcal{P}_{S^o}^{\leq k-1}}$. Comparing the p -right Kan extensions along both paths across the commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{S^o}^{\leq k-1} & \longrightarrow & \mathcal{P}_{S^o}^{\leq k} \\ \downarrow & & \downarrow \\ (\mathcal{P}_{S^o}^{\leq k-1})^\triangleleft & \longrightarrow & (\mathcal{P}_{S^o}^{\leq k})^\triangleleft \end{array}$$

gives the induction step, and thence the result. \square

Corollary 6. Retaining the notation of Lemma 5: any nontrivial subcube of ρ is a p -limit diagram. That is, if U_1 and U_2 are two subsets of S^o with $U_1 \subseteq U_2$ and $|U_2 \setminus U_1| > 1$, then the restriction of ρ to the subposet \mathcal{P}_{U_1, U_2} spanned by those subsets V of S^o with $U_1 \subseteq V \subseteq U_2$ is a p -limit diagram.

Proof. By restricting to \mathcal{P}_{U_2} , we may assume $U_2 = S^o$. Let

$$\mathcal{P}'_{U_1, U_2} = \mathcal{P}_{U_1, U_2} \setminus U_2$$

and let \mathcal{Q} be the closure of \mathcal{P}'_{U_1, U_2} under downward inclusion. Then \mathcal{Q} contains $\mathcal{P}_{S^o}^{\leq 1}$, so by the above discussion, ρ is p -right Kan extended from \mathcal{Q} . But \mathcal{P}'_{U_1, U_2} is coinital in $\mathcal{Q} = \mathcal{Q}_{U_2 /}$, so we're good. \square

Proof of Proposition 3. Now let \mathcal{O}^\otimes be any ∞ -operad, and let $F : (\mathcal{F}_*)_{\langle 1 \rangle /} \rightarrow \mathcal{O}^\otimes$ be a morphism of ∞ -preoperads. Consider the diagram

$$\begin{array}{ccc} (\mathcal{F}_*)_{\langle 1 \rangle /} & \xrightarrow{F} & \mathcal{O}^\otimes \\ \downarrow \phi & \nearrow \phi_* F & \downarrow p \\ \mathcal{CM}^\otimes & \longrightarrow & \mathcal{F}_* \end{array}$$

We claim that the \mathcal{F}_* -relative right Kan extension $\phi_* F$ along the dotted line exists [Lur09, §4.3.2]. Indeed, let (S, U) be an object of \mathcal{CM}^\otimes , and let $\mathcal{Q}_{(S, U)}$ be the subposet of \mathcal{P}_{S° spanned by subsets T such that

- $S^\circ \setminus U \subseteq T$, and
- $|T \cap U| \leq 1$.

Then the natural map

$$j_{(S, U)} : \mathcal{Q}_{(S, U)}^\triangleleft \rightarrow \mathcal{CM}^\otimes$$

which takes all edges of $\mathcal{Q}_{(S, U)}^\triangleleft$ to inert edges of \mathcal{CM}^\otimes gives rise to a map

$$k_{(S, U)} : \mathcal{Q}_{(S, U)} \rightarrow (\mathcal{F}_*)_{\langle 1 \rangle /} \times_{\mathcal{CM}^\otimes} (\mathcal{CM}^\otimes)_{(S, U) /}$$

and $k_{(S, U)}$ is easily observed to be coinitial. Thus it suffices to show that $F \circ k_{(S, U)}$ admits a p -limit. But $F \circ k_{(S, U)}$ can be embedded, up to equivalence, into the cube of inert edges

$$\rho : \mathcal{P}_{S^\circ} \rightarrow \mathcal{O}^\otimes$$

such that

$$\rho_{S^\circ} = \prod_U^p F(\langle 1 \rangle, \{1\}) \times \prod_{S^\circ \setminus U}^p F(\langle 1 \rangle, \emptyset)$$

where \prod^p denotes a product relative to p . By Corollary 6 together with the induction argument used in the proof of Lemma 5, we see that ρ_{S° is a p -limit of $F \circ k_{(S, U)}$. So $\phi_* F$ exists [Lur09, Lemma 4.3.2.13], and it is clear that $\phi_* F$ is a morphism of ∞ -operads. Since any morphism of preoperads from $(\mathcal{F}_*)_{\langle 1 \rangle /}$ to an ∞ -operad extends over \mathcal{CM}^\otimes , ϕ must be a trivial cofibration. \square

Now we'll relate our construction to Lurie's category of modules. Let \mathbf{C}^\otimes be an ∞ -operad. Employing the notation of [Lur12, §3.3.3], we define

$$\mathbf{Mod}(\mathbf{C}) := \mathbf{Mod}^{\mathcal{F}_*}(\mathbf{C})_{\langle 1 \rangle}^\otimes$$

with analogous definitions of $\widetilde{\mathbf{Mod}}(\mathbf{C})$ and $\overline{\mathbf{Mod}}(\mathbf{C})$.

Proposition 7. There is an equivalence of ∞ -categories

$$\mathbf{Mod}(\mathbf{C}) \simeq \mathrm{Fun}^\otimes(\mathcal{CM}^\otimes, \mathbf{C}^\otimes).$$

Proof. Let X be a simplicial set equipped with the constant map $1_X : X \rightarrow \mathcal{F}_*$ with image $\langle 1 \rangle$. One then has set bijections

$$\mathrm{Hom}(X, \widetilde{\mathbf{Mod}}(\mathbf{C})) \cong \mathrm{Hom}_{\mathcal{F}_*}(X \times (\mathcal{F}_*)_{\langle 1 \rangle /}, \mathbf{C}^\otimes)$$

and

$$\mathrm{Hom}(X, \overline{\mathbf{Mod}}(\mathbf{C})) \cong \mathrm{Hom}^\otimes(X \times (\mathcal{F}_*)_{\langle 1 \rangle /}, \mathbf{C}^\otimes)$$

where Hom^\otimes denotes the set of ∞ -preoperad maps; this is to say that we have an isomorphism of categories between $\overline{\mathbf{Mod}}(\mathbf{C})$ and the category $\mathrm{Fun}^\otimes((\mathcal{F}_*)_{\langle 1 \rangle /}, \mathbf{C}^\otimes)$ of ∞ -preoperad maps from $(\mathcal{F}_*)_{\langle 1 \rangle /}$ to \mathbf{C}^\otimes . Moreover, when $\mathcal{O}^\otimes = \mathcal{F}_*$, the trivial

Kan fibration θ of [Lur12, Lemma 3.3.3.3] is an isomorphism, so there is no difference between $\overline{\mathbf{Mod}}(\mathbf{C})$ and $\mathbf{Mod}(\mathbf{C})$.

Finally, we claim that the restriction map

$$\phi^* : \mathrm{Fun}^\otimes(\mathcal{CM}^\otimes, \mathbf{C}^\otimes) \rightarrow \mathrm{Fun}^\otimes((\mathcal{F}_*)_{\langle 1 \rangle /}, \mathbf{C}^\otimes)$$

is a trivial Kan fibration. Noting that the categorical pattern \mathfrak{P}_0 of [Lur12, Lemma B.1.13] serves as a unit for the product of categorical patterns, we deduce from [Lur12, Remark B.2.5] that the product map

$$\mathcal{PO} \times \mathrm{Set}_\Delta^+ \rightarrow \mathcal{PO}$$

is a left Quillen bifunctor. This means, in particular, that for each n , the morphism of ∞ -preoperads

$$((\mathcal{F}_*)_{\langle 1 \rangle /} \times (\Delta^n)^b) \cup_{(\mathcal{F}_*)_{\langle 1 \rangle /} \times (\partial \Delta^n)^b} (\mathcal{CM}^\otimes \times (\partial \Delta^n)^b) \rightarrow \mathcal{CM}^\otimes \times (\Delta^n)^b$$

is a trivial cofibration in \mathcal{PO} , which gives the result. \square

We now wish to characterize the symmetric monoidal envelope of \mathcal{CM}^\otimes .

Definition 8. Let \mathcal{F}^+ be the category whose objects are pairs (S, U) , with S a finite set and $U \subseteq S$ a subset, and in which a morphism from $(S, U) \rightarrow (T, V)$ is a morphism $f : S \rightarrow T$ of finite sets inducing a bijection $f|_U : U \cong V$. The disjoint union (which, mind you, is definitely not a coproduct) makes \mathcal{F}^+ into a symmetric monoidal category $(\mathcal{F}^+)^\amalg$.

Proposition 9. The symmetric monoidal envelope $\mathrm{Env}(\mathcal{CM}^\otimes)$ is isomorphic to $(\mathcal{F}^+)^\amalg$.

Proof. We briefly sketch the proof, which is a routine 1-categorical exercise. By definition, the symmetric monoidal envelope of $\mathrm{Env}(\mathcal{CM}^\otimes)$ has objects

$$(S, U, f : S^o \rightarrow T)$$

where $T \in \mathcal{F}$. Our isomorphism will map this object to the object

$$(T, (f^{-1}(t), f^{-1}(t) \cap U)_{t \in T})$$

of $(\mathcal{F}^+)^\amalg$. \square

Corollary 10. A \mathcal{CM}^\otimes -algebra parametrizing a commutative monoid E and a module M in \mathbf{C}^\otimes gives rise to a functor $A_{E,M} : \mathcal{F}_* \rightarrow \mathbf{C}$ such that

$$A_{E,M}(S) \simeq E^{\otimes S^o} \otimes M.$$

Proof. We construct $A_{E,M}$ by embedding \mathcal{F}_* as the full subcategory of \mathcal{F}^+ consisting of objects (S, U) for which $|U| = 1$. \square

REFERENCES

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